

Real spinors and real Dirac equation

Václav Zatloukal

FNSPE, Czech Technical University in Prague

[www.zatlovac.eu]

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Motivation

$$i\hbar\partial_t\psi = \hat{H}\psi \quad , \quad (i\gamma^\mu\partial_\mu - m)\psi = 0 \quad , \quad \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}S[\phi, \mathcal{J}]\right)$$

Find geometric meaning of i , the unit imaginary of quantum theory.

Make use of a rich geometric structure of real Clifford algebra
(aka. *geometric algebra*).

Following:

D. Hestenes, *Spacetime Physics with Geometric Algebra*, Am. J. Phys. **71** 6, (2003)

C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, CUP (2007)

- Real Clifford algebras:
rotations and real spinors in $2D$, $3D$ and $3 + 1D$
- Dirac equation:
matrix form vs. real form, gauge fields

V. Z., *Real spinors and real Dirac equation*, arXiv:1908.04590 (2019)

Real Clifford algebra

Real vector space $V = \text{span}\{e_1, \dots, e_n\}$
with quadratic form $Q(e_j) = \pm 1$ of signature (p, q)

→ Real Clifford algebra: “freest” associative algebra generated by V ,
subject to $a^2 = Q(a)$, $\forall a \in V$

$$\mathcal{Cl}(V^{p,q}) = \text{span}\{1, e_i, e_i e_j (i < j), \dots, e_1 e_2 \dots e_n\} \quad (\dim = 2^n)$$

Clifford (geometric) product: $ab = a \cdot b + a \wedge b \quad (a, b \in V)$

$a \cdot b = \frac{1}{2}(ab + ba)$ is a scalar (by $(a + b)^2 = a^2 + b^2 + ab + ba$)

$a \wedge b = \frac{1}{2}(ab - ba)$ is a bivector

Matrix representation: $e_1, \dots, e_n \leftrightarrow \gamma$ -matrices

Geometric representation: r -vectors $\leftrightarrow r$ -dim. parallelograms

D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987).

Even subalgebra: $\mathcal{Cl}_{\text{even}}(V^{p,q}) = \text{span}\{1, e_i e_j (i < j), \dots\} \quad (\dim = 2^{n-1})$

2D: Rotations and complex numbers

Euclidean plane $E^2 = \text{span}\{e_1, e_2\}$

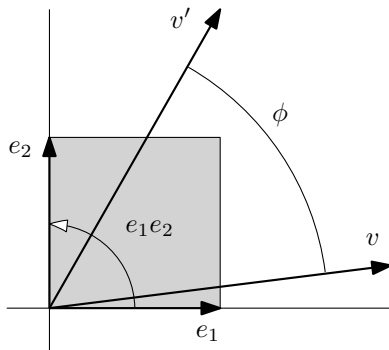
$$[e_1^2 = e_2^2 = 1, e_1 \cdot e_2 = 0]$$

→ Clifford algebra

$$\mathcal{Cl}(E^2) = \text{span}\{1, e_1, e_2, e_1 e_2\}$$

Even subalgebra

$$\mathcal{Cl}_{\text{even}}(E^2) = \text{span}\{1, e_1 e_2\}$$



'Complex numbers': $x + y e_1 e_2 = r e^{\phi e_1 e_2}$ $[(e_1 e_2)^2 = -1]$

Rotation by angle ϕ :

$$v \mapsto v e^{\phi e_1 e_2} = e^{-\frac{\phi}{2} e_1 e_2} v e^{\frac{\phi}{2} e_1 e_2} \quad (1)$$

3D: Rotations and real spinors

Euclidean space $E^3 = \text{span}\{e_1, e_2, e_3\}$

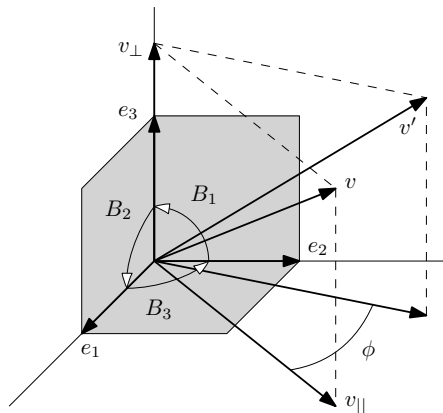
$$[e_i \cdot e_j = \delta_{ij}]$$

→ Clifford algebra $\mathcal{Cl}(E^3) = \text{span}\{1, e_1, e_2, e_3, B_1, B_2, B_3, e_1 e_2 e_3\}$

$$[B_i = \frac{\varepsilon_{ijk}}{2} e_j e_k \leftrightarrow \text{quaternionic units}]$$

Rotation in B -plane by angle $|B|$:

$$v \mapsto U v U^{-1}, \quad U = e^{-B/2} \quad (2)$$



Rotors U form group $\text{Spin}(3)$ (double-cover of $\text{SO}(3)$)

Spinors: $\psi \in \mathcal{Cl}_{\text{even}}(E^3) = \text{span}\{1, B_1, B_2, B_3\} = \text{span}\{e^{-B/2} \mid B \in \mathcal{Cl}_2\}$

Spinorial representation: $U\psi$ (cf. $U\psi U^{-1} \Leftarrow v_1 \dots v_r \mapsto Uv_1 \dots v_r U^{-1}$)

3D: Real spinors and Pauli spinors

Real spinor $\psi = \alpha_0 + \alpha_i B_i = \sqrt{\rho} R$ (rotor with magnitude)

(4 *real* components $\leftrightarrow$ 2 *complex* numbers)

→ **Pauli spinor**

$$|\psi\rangle \equiv \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2, \quad \begin{aligned} z_0 &= \langle \psi (1 - i B_3) \rangle \\ z_1 &= \langle B_2 \psi (1 - i B_3) \rangle \end{aligned} \quad (3)$$

Scalar part $\langle . \rangle$, $\langle AB \rangle = \langle BA \rangle$ (cf. $\text{tr}(.)$ in matrix representation)

Algebraic operations on $\psi \leftrightarrow$ Matrix operations on $|\psi\rangle$:

$$\begin{aligned} |B_j \psi\rangle &= i \hat{\sigma}_j |\psi\rangle \\ |\psi B_1\rangle &= \hat{\sigma}_2 |\psi\rangle^* \\ |\psi B_2\rangle &= i \hat{\sigma}_2 |\psi\rangle^* \\ |\psi B_3\rangle &= i |\psi\rangle \end{aligned} \quad (4)$$

(Bivectors $\{B_j/2\}$ form a representation of $su(2)$: $[B_i, B_j] = -2\epsilon_{ijk} B_k$)

3+1D: Lorentz transformations and real spinors

Minkowski spacetime $E^{1,3} = \text{span}\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$

$$\rightarrow \mathcal{Cl}(E^{1,3}) = \text{span}\{1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu I, I\} , I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \quad (\text{dim } 16)$$

Lorentz transformation (proper, orthochronous):

$$v \mapsto U v \tilde{U} \quad , \quad (a \dots b)^\sim = b \dots a \quad (5)$$

(Typically: $U = e^{-B/2}$, where $B = \frac{1}{2} B^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$)

Spinors: $\Psi \in \mathcal{Cl}_{\text{even}}(E^{1,3}) = \text{span}\{1, \gamma_\mu \wedge \gamma_\nu, I\} = \text{span}\{e^{-B/2}\}$ (dim 8)

(For $V^{p,q}$ define spinors as $\mathcal{Cl}_{\text{even}}(V^{p,q})$ — smallest linear space containing all rotors.)

3+1D: Real spinors and Dirac spinors

Real spinor Ψ has 8 *real* components $\leftrightarrow$ 4 *complex* numbers

$\rightarrow$ **Dirac spinor**

$$|\Psi\rangle \equiv \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^4, \quad \begin{aligned} z_0 &= \langle \Psi(1 - i\gamma_2\gamma_1) \rangle \\ z_1 &= \langle \gamma_1\gamma_3\Psi(1 - i\gamma_2\gamma_1) \rangle \\ z_2 &= \langle \gamma_3\gamma_0\Psi(1 - i\gamma_2\gamma_1) \rangle \\ z_3 &= \langle \gamma_1\gamma_0\Psi(1 - i\gamma_2\gamma_1) \rangle \end{aligned} \quad (6)$$

Algebraic operations on $\Psi \leftrightarrow$ Matrix operations on $|\Psi\rangle$:

$$|\gamma_\mu\Psi\gamma_0\rangle = \hat{\gamma}_\mu|\Psi\rangle$$

$$|\gamma_\mu\gamma_\nu\Psi\rangle = \hat{\gamma}_\mu\hat{\gamma}_\nu|\Psi\rangle$$

$$|\Psi\gamma_2\gamma_1\rangle = i|\Psi\rangle, \dots \quad (7)$$

where $\hat{\gamma}_\mu$ are Dirac γ -matrices in *standard* representation:

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \hat{\gamma}_1 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}, \quad \hat{\gamma}_2 = \begin{pmatrix} & & i & \\ & -i & & \\ & & & i \\ i & & & \end{pmatrix}, \quad \hat{\gamma}_3 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & 1 \end{pmatrix}$$

Dirac equation - flat spacetime

Matrix Dirac equation (for Dirac spinors) (Dirac 1928):

$$i\hat{\gamma}^{\mu}\partial_{\mu}|\Psi\rangle - m|\Psi\rangle = 0 \quad (8)$$

Real Dirac equation (for real spinors) (Hestenes 1966):

$$\gamma^{\mu}(\partial_{\mu}\Psi)\gamma_0\gamma_2\gamma_1 - m\Psi = 0 \quad (9)$$

Lorentz invariance: $x'^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu}$, where $(\Lambda^{-1})^{\nu}_{\mu}\gamma^{\mu} = U\gamma^{\nu}\tilde{U}$

→ Transformation

$$\gamma'_{\mu} = \gamma_{\mu} \quad , \quad \Psi'(x') = U\Psi(x) \quad (10)$$

does not respect that real spinors Ψ are polynomials in γ_{μ} .

Dirac equation - curved spacetime

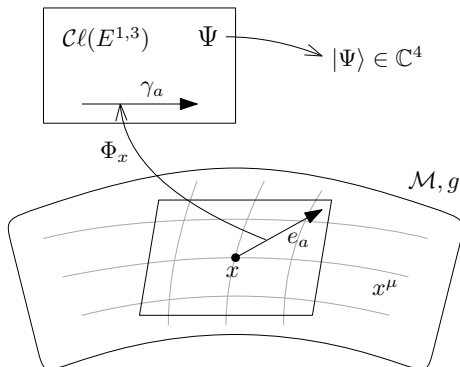
Manifold $\mathcal{M}$ with metric $g_{\mu\nu}$

Tetrad field e_a^μ

Φ_x : tangent space $\rightarrow$ flat space

Local Lorentz (gauge) transf.:

$$\gamma'_a = \Phi'_x(e_a) = U \gamma_a \tilde{U}, \quad \Psi' = U \Psi \tilde{U}$$



Real Dirac equation in curved spacetime:

$$\gamma^a e_a^\mu (D_\mu \Psi) \gamma_0 \gamma_2 \gamma_1 - m \Psi = 0,$$

$$\hookrightarrow D_\mu \Psi = \partial_\mu \Psi - \omega_\mu \Psi + \Psi \mathcal{A}_\mu \quad (11)$$

Gauge fields ω_μ , $\mathcal{A}_\mu$ are bivector-valued 1-forms that transform as:

$$\omega'_\mu = U \omega_\mu \tilde{U} + (\partial_\mu U) \tilde{U}, \quad \mathcal{A}'_\mu = U \mathcal{A}_\mu \tilde{U} + (\partial_\mu U) \tilde{U}$$

Dirac equation - matrix form

The operation $| \rangle$ takes complex components:

$$i\hat{\gamma}^a e_a^\mu |D_\mu \Psi\rangle - m|\Psi\rangle = 0 \quad (12)$$

where the covariant derivative translates as

$$|D_\mu \Psi\rangle = \partial_\mu |\Psi\rangle + \frac{\omega_\mu^{ab}}{2} \hat{\gamma}_a \hat{\gamma}_b |\Psi\rangle - \frac{\mathcal{A}_\mu^{ab}}{2} |\Psi \gamma_a \gamma_b\rangle \quad (13)$$

$$|\Psi \gamma_2 \gamma_1\rangle = i|\Psi\rangle$$

$$\rightarrow \mathcal{A}_\mu^{12} = qA_\mu \dots \text{elmag. potential}$$

$$|\Psi \gamma_3 \gamma_0\rangle = \hat{\gamma}_5 |\Psi\rangle$$

(D. Hestenes, arXiv:0807.0060v1

$$|\Psi \gamma_1 \gamma_0\rangle = -i\hat{\gamma}_2 |\Psi\rangle^*$$

– about electroweak theory)

$$|\Psi \gamma_2 \gamma_0\rangle = \hat{\gamma}_2 |\Psi\rangle^*$$

$$|\Psi \gamma_3 \gamma_2\rangle = -\hat{\gamma}_2 \hat{\gamma}_5 |\Psi\rangle^*$$

$$|\Psi \gamma_1 \gamma_3\rangle = -i\hat{\gamma}_2 \hat{\gamma}_5 |\Psi\rangle^*$$

Here, $\hat{\gamma}_5 = i\hat{\gamma}^0 \hat{\gamma}^1 \hat{\gamma}^2 \hat{\gamma}^3$,

and $*$ denotes component-wise complex conjugation.

Further remarks

- Gauge field strength:

$$[D_\mu, D_\nu]\Psi = -\mathcal{R}_{\mu\nu}\Psi + \Psi\mathcal{F}_{\mu\nu} \quad (14)$$

where

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - [\omega_\mu, \omega_\nu] \\ \mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - [\mathcal{A}_\mu, \mathcal{A}_\nu]\end{aligned} \quad (15)$$

- Yang-Mills equations (flat spacetime: $e_a^\mu = \delta_a^\mu$, $\omega_\mu = 0$):

$$\partial_\mu\mathcal{F}^{\mu\nu} - [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0 \quad (16)$$

- For invertible real spinors Ψ ($\Psi = \sqrt{\rho} e^{I\beta/2} R$),

$$\gamma^\mu(\partial_\mu\Psi)\gamma_2\gamma_1\Psi^{-1} + q\gamma^\mu A_\mu - m\Psi\gamma_0\Psi^{-1} = 0 \quad (17)$$

allows to express A_μ in terms of Ψ and $\partial_\mu\Psi$.

(See also C. J. Radford, J. Math. Phys. **37**, 4418 (1996), arXiv:hep-th/9510065)

Summary and discussion

- Spinors ψ and Ψ as even elements of real Clifford algebra (relation with Pauli and Dirac spinors $|\psi\rangle$ and $|\Psi\rangle$)
- Interpreting i as a spatial bivector (B_3 or $\gamma_2\gamma_1$)
- In principle, “ $i(x)$ ” can vary throughout spacetime
- Non-Abelian connection $\mathcal{A}_\mu$ acting on the right of Ψ , (generalizing the electromagnetic potential A_μ)

Thank you for your attention.

V. Z., Real spinors and real Dirac equation, arXiv:1908.04590 (2019)

3D: Relation with Pauli spinors

Hermitian conjugate: $\langle \Psi | = (z_0^*, z_1^*)$

$\rightarrow \text{Re}\langle \Phi | \Psi \rangle = \langle \tilde{\Phi} | \Psi \rangle$ (reversion: $\widetilde{a \dots b} = b \dots a$, e.g. $\tilde{B}_j = -B_j$)

$$\langle \Phi | \Psi \rangle = \text{Re}\langle \Phi | \Psi \rangle - i \text{Re}(i\langle \Phi | \Psi \rangle) = \langle \tilde{\Phi} | \Psi (1 - i B_3) \rangle \quad (18)$$

Spin “vector”:

$$\begin{aligned} \langle \Psi | \hat{\sigma}_j | \Psi \rangle &= \langle \Psi | i \hat{\sigma}_j (-i) | \Psi \rangle = -\langle \Psi | B_j \Psi B_3 \rangle = \langle \tilde{\Psi} \tilde{B}_j \Psi B_3 \rangle \\ &= \langle \tilde{B}_j \Psi B_3 \tilde{\Psi} \rangle \end{aligned} \quad (19)$$

$\rightarrow$ Bivector $\Psi B_3 \tilde{\Psi} = \rho R B_3 R^{-1}$ (B_3 rotated by R and stretched by ρ)

$$\Psi B_3 \tilde{\Psi} = \langle \Psi | \hat{\sigma}_j | \Psi \rangle B_j \quad (20)$$

3+1D: Dirac current and other observables

Hermitian conjugate: $\langle \Psi | = (z_0^*, z_1^*, z_2^*, z_3^*) \rightarrow \text{Re} \langle \Phi | \Psi \rangle = \langle \tilde{\Phi} \gamma_0 \Psi \gamma_0 \rangle$

Dirac conjugate: $\langle \bar{\Psi} | = \langle \Psi | \hat{\gamma}_0 \rightarrow \text{Re} \langle \bar{\Phi} | \Psi \rangle = \text{Re} \langle \Phi | \gamma_0 \Psi \gamma_0 \rangle = \langle \tilde{\Phi} \Psi \rangle$

$$\Rightarrow \langle \bar{\Phi} | \Psi \rangle = \langle \tilde{\Phi} \Psi (1 - i\gamma_2 \gamma_1) \rangle \quad (21)$$

Dirac current:

$$\langle \bar{\Psi} | \hat{\gamma}_\mu | \Psi \rangle = \langle \bar{\Psi} | \gamma_\mu \Psi \gamma_0 \rangle = \langle \tilde{\Psi} \gamma_\mu \Psi \gamma_0 \rangle = \langle \gamma_\mu \Psi \gamma_0 \tilde{\Psi} \rangle \quad (22)$$

(Canonical form $\Psi = \sqrt{\rho} e^{I\beta/2} R$ & $I\gamma_\mu = -\gamma_\mu I$
 $\Rightarrow \Psi \gamma_0 \tilde{\Psi} = \rho R \gamma_0 \tilde{R}$ is a vector rotated by R and stretched by ρ)

$$\Psi \gamma_0 \tilde{\Psi} = \langle \bar{\Psi} | \hat{\gamma}_\mu | \Psi \rangle \gamma^\mu \quad (23)$$

Other observables:

$$\begin{aligned} \langle \bar{\Psi} | \frac{i}{2} [\hat{\gamma}^\mu, \hat{\gamma}^\nu] | \Psi \rangle &= (\gamma^\mu \wedge \gamma^\nu) \cdot (\Psi \gamma_2 \gamma_1 \tilde{\Psi}) \quad , \quad \langle \bar{\Psi} | \Psi \rangle = \langle \tilde{\Psi} \Psi \rangle \\ \langle \bar{\Psi} | \hat{\gamma}^\mu \hat{\gamma}_5 | \Psi \rangle &= \gamma^\mu \cdot (\Psi \gamma_3 \tilde{\Psi}) \quad , \quad \langle \bar{\Psi} | i \hat{\gamma}_5 | \Psi \rangle = \langle \Psi \tilde{\Psi} I \rangle \end{aligned}$$